A partial Laplacian as an infinitesimal generator on the Wasserstein space

YT. Chow and W. Gangbo
NIPS 17 - OTML

December 9, 2017

- Let $\mathcal{P}_2(\mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d$, of finite second moments

- Let $\mathcal{P}_2(\mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d$, of finite second moments
- Goal 1:

- Let $\mathcal{P}_2(\mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d$, of finite second moments
- Goal 1:

  Define Brownian motions on $\mathcal{P}_2(\mathbb{R}^d)$

-

- Let $\mathcal{P}_2(\mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d$, of finite second moments
- Goal 1:

  Define Brownian motions on $\mathcal{P}_2(\mathbb{R}^d)$

-
-

- Let $\mathcal{P}_2(\mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d$, of finite second moments
- Goal 1:

  Define Brownian motions on $\mathcal{P}_2(\mathbb{R}^d)$

-
-
- Goal 2:

- Let $\mathcal{P}_2(\mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d$, of finite second moments
- Goal 1:

  Define Brownian motions on $\mathcal{P}_2(\mathbb{R}^d)$
-
-
- Goal 2:

  Convince you we have the right definition
-

- Let $\mathcal{P}_2(\mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d$, of finite second moments
- Goal 1:

  Define Brownian motions on $\mathcal{P}_2(\mathbb{R}^d)$
-
-
- Goal 2:

  Convince you we have the right definition
-
-

- Let $\mathcal{P}_2(\mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d$, of finite second moments
- Goal 1:

  Define Brownian motions on $\mathcal{P}_2(\mathbb{R}^d)$
-
-
- Goal 2:

  Convince you we have the right definition
-
-
- Goal 3:

- Let $\mathcal{P}_2(\mathbb{R}^d)$ be the set of probability measures on $\mathbb{R}^d$, of finite second moments
- Goal 1:

  Define Brownian motions on $\mathcal{P}_2(\mathbb{R}^d)$
-
-
- Goal 2:

  Convince you we have the right definition
-
-
- Goal 3:

  Convince you of their useful
-

Goal 3: Example

- For each integer $n \geq 1$ assume we solve

$$\partial_t U^n(t,x) = \sum_{j=1}^{n} \triangle_{x_j} U^n(t,x), \qquad t > 0, \quad x \in \mathbb{R}^d \times \cdots \times \mathbb{R}^d \quad (1)$$

Goal 3: Example

- For each integer $n \geq 1$ assume we solve

$$\partial_t U^n(t,x) = \sum_{j=1}^{n} \triangle_{x_j} U^n(t,x), \qquad t > 0, \quad x \in \mathbb{R}^d \times \cdots \times \mathbb{R}^d \quad (1)$$

- Assume $U^n(t,\cdot)$ is invariant under permutation of the x_i :

$$U^n(t, x_1, \cdots, x_{n-1}, x_n) = U^n(t, x_1, \cdots, x_n, x_{n-1})$$

Goal 3: Example

- For each integer $n \geq 1$ assume we solve

$$\partial_t U^n(t,x) = \sum_{j=1}^{n} \triangle_{x_j} U^n(t,x), \qquad t > 0, \quad x \in \mathbb{R}^d \times \cdots \times \mathbb{R}^d \quad (1)$$

- Assume $U^n(t,\cdot)$ is invariant under permutation of the x_i :

$$U^n(t, x_1, \cdots, x_{n-1}, x_n) = U^n(t, x_1, \cdots, x_n, x_{n-1})$$

- We want to make sense of

$$U^n(t,\cdot) \to U, \qquad \text{as} \quad n \to \infty$$

Goal 3: Example

- For each integer $n \geq 1$ assume we solve

$$\partial_t U^n(t,x) = \sum_{j=1}^{n} \triangle_{x_j} U^n(t,x), \qquad t > 0, \quad x \in \mathbb{R}^d \times \cdots \times \mathbb{R}^d \quad (1)$$

- Assume $U^n(t,\cdot)$ is invariant under permutation of the x_i :

$$U^n(t, x_1, \cdots, x_{n-1}, x_n) = U^n(t, x_1, \cdots, x_n, x_{n-1})$$

- We want to make sense of

$$U^n(t,\cdot) \to U, \qquad \text{as} \quad n \to \infty$$

- and check what equation U satisfies:

$$\partial_t U = \sum_{j=1}^{\infty} \triangle_{x_j} U?$$

Review: geometry and stochastic

- Let $\mathbb{M}$ be a finite dimensional euclidean space (e.g. $\mathbb{M} = \mathbb{R}^d$).

Review: geometry and stochastic

- Let $\mathbb{M}$ be a finite dimensional euclidean space (e.g. $\mathbb{M} = \mathbb{R}^d$).
- For $x \in \mathbb{M}$, let $T_x\mathbb{M}$ denote the tangent space at x

Review: geometry and stochastic

- Let $\mathbb{M}$ be a finite dimensional euclidean space (e.g. $\mathbb{M} = \mathbb{R}^d$).
- For $x \in \mathbb{M}$, let $T_x\mathbb{M}$ denote the tangent space at x
- The matrix of the second derivatives of $g \in C^2(\mathbb{M})$ define a bilinear form

$$\text{hess}\, g(x) : T_x\mathbb{M} \times T_x\mathbb{M} \to \mathbb{R} :$$

$$\text{hess}\, g(x)(v, v) = \frac{d^2}{dt^2} g(x + tv)\Big|_{t=0}$$

Review: geometry and stochastic

- Let $\mathbb{M}$ be a finite dimensional euclidean space (e.g. $\mathbb{M} = \mathbb{R}^d$).
- For $x \in \mathbb{M}$, let $T_x\mathbb{M}$ denote the tangent space at x
- The matrix of the second derivatives of $g \in C^2(\mathbb{M})$ define a bilinear form

$$\mathrm{hess}\, g(x) : T_x\mathbb{M} \times T_x\mathbb{M} \to \mathbb{R} :$$

$$\mathrm{hess}\, g(x)(v, v) = \frac{d^2}{dt^2} g(x + tv)\Big|_{t=0}$$

- It has been obtained by the 2nd derivatives of g along the geodesic $t \to x + tv$

Review: geometry and stochastic

- Let $\mathbb{M}$ be a finite dimensional euclidean space (e.g. $\mathbb{M} = \mathbb{R}^d$).
- For $x \in \mathbb{M}$, let $T_x\mathbb{M}$ denote the tangent space at x
- The matrix of the second derivatives of $g \in C^2(\mathbb{M})$ define a bilinear form

$$\mathrm{hess}\, g(x) : T_x\mathbb{M} \times T_x\mathbb{M} \to \mathbb{R} :$$

$$\mathrm{hess}\, g(x)(v, v) = \frac{d^2}{dt^2} g(x + tv)\Big|_{t=0}$$

- It has been obtained by the 2nd derivatives of g along the geodesic $t \to x + tv$
- We have

$$\triangle g(x) = \sum_{j=1}^{d} \mathrm{hess}\, g(x)(v_j, v_j)$$

Review: geometry and stochastic

- Let $\mathbb{M}$ be a finite dimensional euclidean space (e.g. $\mathbb{M} = \mathbb{R}^d$).
- For $x \in \mathbb{M}$, let $T_x\mathbb{M}$ denote the tangent space at x
- The matrix of the second derivatives of $g \in C^2(\mathbb{M})$ define a bilinear form

$$\text{hess } g(x) : T_x\mathbb{M} \times T_x\mathbb{M} \to \mathbb{R} :$$

$$\text{hess } g(x)(v, v) = \frac{d^2}{dt^2} g(x + tv)\Big|_{t=0}$$

- It has been obtained by the 2nd derivatives of g along the geodesic $t \to x + tv$
- We have

$$\triangle g(x) = \sum_{j=1}^{d} \text{hess } g(x)(v_j, v_j)$$

- Here $\{v_j\}_{j=1}^d$ is any orthonormal basis of $T_x\mathbb{M}$

Review: geometry and stochastic

- Let $(W_t)_{t\geq 0}$ be a Brownian motion on $\mathbb{R}^d$

Review: geometry and stochastic

- Let $(W_t)_{t\geq 0}$ be a Brownian motion on $\mathbb{R}^d$
- We have

$$\triangle g(x) = \lim_{t\to 0} \frac{\mathbb{E}(g(x+\sqrt{2}W_t)) - g(x)}{t}$$

- Let $(W_t)_{t\geq 0}$ be a Brownian motion on $\mathbb{R}^d$
- We have

$$\triangle g(x) = \lim_{t\to 0} \frac{\mathbb{E}(g(x+\sqrt{2}W_t)) - g(x)}{t}$$

- This is related to Feynman-Kac formula: if we set

$$u(t,x) = \mathbb{E}\Big(g(x+\sqrt{2}W_t)\Big)$$

Review: geometry and stochastic

- Let $(W_t)_{t\geq 0}$ be a Brownian motion on $\mathbb{R}^d$
- We have

$$\triangle g(x) = \lim_{t\to 0} \frac{\mathbb{E}(g(x+\sqrt{2}W_t)) - g(x)}{t}$$

- This is related to Feynman-Kac formula: if we set

$$u(t,x) = \mathbb{E}\Big(g(x+\sqrt{2}W_t)\Big)$$

- then

$$\partial_t u = \triangle u, \qquad u(0,\cdot) = g$$

EXCURSION TO WASSERSTEIN GEOMETRY

- Our first goal is to extend the above considerations to the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$.

EXCURSION TO WASSERSTEIN GEOMETRY

- Our first goal is to extend the above considerations to the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$.
- This includes computing hessians on $\mathcal{P}_2(\mathbb{R}^d)$

EXCURSION TO WASSERSTEIN GEOMETRY

- Our first goal is to extend the above considerations to the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$.
- This includes computing hessians on $\mathcal{P}_2(\mathbb{R}^d)$
- The first annoying complication to describe hessians is

EXCURSION TO WASSERSTEIN GEOMETRY

- Our first goal is to extend the above considerations to the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$.
- This includes computing hessians on $\mathcal{P}_2(\mathbb{R}^d)$
- The first annoying complication to describe hessians is
- $\mathcal{P}_2(\mathbb{R}^d)$ is not a euclidean space

EXCURSION TO WASSERSTEIN GEOMETRY

- Our first goal is to extend the above considerations to the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$.
- This includes computing hessians on $\mathcal{P}_2(\mathbb{R}^d)$
- The first annoying complication to describe hessians is
- $\mathcal{P}_2(\mathbb{R}^d)$ is not a euclidean space
- Life would be simpler on a Hilbert space such as

$$\mathbb{H} := L^2(\Omega, \mathbb{R}^d), \qquad \Omega := (0,1)^d \subset \mathbb{R}^d.$$

EXCURSION TO WASSERSTEIN GEOMETRY

- Our first goal is to extend the above considerations to the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$.
- This includes computing hessians on $\mathcal{P}_2(\mathbb{R}^d)$
- The first annoying complication to describe hessians is
- $\mathcal{P}_2(\mathbb{R}^d)$ is not a euclidean space
- Life would be simpler on a Hilbert space such as

$$\mathbb{H} := L^2\big(\Omega, \mathbb{R}^d\big), \qquad \Omega := (0,1)^d \subset \mathbb{R}^d.$$

- for future studies, we showed (G-Tudorascu):

EXCURSION TO WASSERSTEIN GEOMETRY

- Our first goal is to extend the above considerations to the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$.
- This includes computing hessians on $\mathcal{P}_2(\mathbb{R}^d)$
- The first annoying complication to describe hessians is
- $\mathcal{P}_2(\mathbb{R}^d)$ is not a euclidean space
- Life would be simpler on a Hilbert space such as

$$\mathbb{H} := L^2\big(\Omega, \mathbb{R}^d\big), \qquad \Omega := (0,1)^d \subset \mathbb{R}^d.$$

- for future studies, we showed (G-Tudorascu):
- the differential structure on $\mathcal{P}_2(\mathbb{R}^d)$, by Ambrosio–Gigli–Savaré

EXCURSION TO WASSERSTEIN GEOMETRY

- Our first goal is to extend the above considerations to the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$.
- This includes computing hessians on $\mathcal{P}_2(\mathbb{R}^d)$
- The first annoying complication to describe hessians is
- $\mathcal{P}_2(\mathbb{R}^d)$ is not a euclidean space
- Life would be simpler on a Hilbert space such as

$$\mathbb{H} := L^2\big(\Omega, \mathbb{R}^d\big), \qquad \Omega := (0,1)^d \subset \mathbb{R}^d.$$

- for future studies, we showed (G-Tudorascu):
- the differential structure on $\mathcal{P}_2(\mathbb{R}^d)$, by Ambrosio–Gigli–Savaré
- amount

$$\mathcal{P}_2(\mathbb{R}^d) = \mathbb{H}/\textit{equi relation}$$

- Lift $G : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ to obtain $\bar{G} : \mathbb{H} \to \mathbb{R}$

- Lift $G : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ to obtain $\bar{G} : \mathbb{H} \to \mathbb{R}$
- $$\text{"}\mathrm{Hess}_{\mathcal{P}_2(\mathbb{R}^d)} G = \mathrm{Hess}_{\mathbb{H}} \bar{G}\text{"}$$

- Lift $G : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ to obtain $\bar{G} : \mathbb{H} \to \mathbb{R}$
-
$$\text{"}\mathrm{Hess}_{\mathcal{P}_2(\mathbb{R}^d)} G = \mathrm{Hess}_{\mathbb{H}} \bar{G}\text{"}$$
- Recall: if
$$\mathbf{v} := \nabla \varphi \in \nabla C_c^\infty\Big((-1,1) \times \mathbb{R}^d, \mathbb{R}^d\Big)$$

- Lift $G : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ to obtain $\bar{G} : \mathbb{H} \to \mathbb{R}$
- $$\text{"}\mathrm{Hess}_{\mathcal{P}_2(\mathbb{R}^d)} G = \mathrm{Hess}_{\mathbb{H}} \bar{G}\text{"}$$
- Recall: if
$$\mathbf{v} := \nabla\varphi \in \nabla C_c^\infty\Big((-1,1) \times \mathbb{R}^d, \mathbb{R}^d\Big)$$
- given $m \in \mathcal{P}_2(\mathbb{R}^d)$, we have a geodesic on $\mathcal{P}_2(\mathbb{R}^d)$ starting at m:
$$\partial_t \sigma + \nabla \cdot (\sigma \mathbf{v}) = 0, \qquad \sigma(0) = m$$

- Lift $G : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ to obtain $\bar{G} : \mathbb{H} \to \mathbb{R}$
- $$\text{"}\mathrm{Hess}_{\mathcal{P}_2(\mathbb{R}^d)} G = \mathrm{Hess}_{\mathbb{H}} \bar{G}\text{"}$$
- Recall: if
$$\mathbf{v} := \nabla\varphi \in \nabla C_c^\infty\Big((-1,1) \times \mathbb{R}^d, \mathbb{R}^d\Big)$$
- given $m \in \mathcal{P}_2(\mathbb{R}^d)$, we have a geodesic on $\mathcal{P}_2(\mathbb{R}^d)$ starting at m:
$$\partial_t \sigma + \nabla \cdot \big(\sigma \mathbf{v}\big) = 0, \qquad \sigma(0) = m$$
- The Hessian of $G : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ at m is a bilinear form

- Lift $G : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ to obtain $\bar{G} : \mathbb{H} \to \mathbb{R}$
- $$"\mathrm{Hess}_{\mathcal{P}_2(\mathbb{R}^d)} G = \mathrm{Hess}_{\mathbb{H}} \bar{G}"$$
- Recall: if
$$\mathbf{v} := \nabla\varphi \in \nabla C_c^\infty\Big((-1,1) \times \mathbb{R}^d, \mathbb{R}^d\Big)$$
- given $m \in \mathcal{P}_2(\mathbb{R}^d)$, we have a geodesic on $\mathcal{P}_2(\mathbb{R}^d)$ starting at m:
$$\partial_t \sigma + \nabla \cdot \big(\sigma \mathbf{v}\big) = 0, \qquad \sigma(0) = m$$
- The Hessian of $G : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ at m is a bilinear form
- $$\mathrm{Hess}\, G[m](\xi, \xi) = \frac{d^2}{dt^2} G(\sigma(t))\Big|_{t=0}, \qquad \xi := \mathbf{v}(0, \cdot)$$

- Lift $G : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ to obtain $\bar{G} : \mathbb{H} \to \mathbb{R}$
- $$\text{"}\mathrm{Hess}_{\mathcal{P}_2(\mathbb{R}^d)} G = \mathrm{Hess}_{\mathbb{H}} \bar{G}\text{"}$$
- Recall: if
$$\mathbf{v} := \nabla\varphi \in \nabla C_c^\infty\Big((-1,1) \times \mathbb{R}^d, \mathbb{R}^d\Big)$$
- given $m \in \mathcal{P}_2(\mathbb{R}^d)$, we have a geodesic on $\mathcal{P}_2(\mathbb{R}^d)$ starting at m:
$$\partial_t \sigma + \nabla \cdot (\sigma \mathbf{v}) = 0, \qquad \sigma(0) = m$$
- The Hessian of $G : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ at m is a bilinear form
- $$\mathrm{Hess}\, G[m](\xi, \xi) = \frac{d^2}{dt^2} G(\sigma(t))\Big|_{t=0}, \qquad \xi := \mathbf{v}(0, \cdot)$$
-

EXCURSION IN STOCHASTIC WORLD

- For a while, stop with the above deterministic considerations

EXCURSION IN STOCHASTIC WORLD

- For a while, stop with the above deterministic considerations
- Choose a Brownian motion $(W_t)_{t\geq 0}$ on $\mathbb{R}^d$

EXCURSION IN STOCHASTIC WORLD

- For a while, stop with the above deterministic considerations
- Choose a Brownian motion $(W_t)_{t \geq 0}$ on $\mathbb{R}^d$
- Lift it to $\mathcal{P}_2(\mathbb{R}^d)$ as follows

EXCURSION IN STOCHASTIC WORLD

- For a while, stop with the above deterministic considerations
- Choose a Brownian motion $(W_t)_{t\geq 0}$ on $\mathbb{R}^d$
- Lift it to $\mathcal{P}_2(\mathbb{R}^d)$ as follows
- $$t \to \mathbb{B}_t^m := \left(\mathbf{id} + \sqrt{2}W_t\right)_{\#} m$$

EXCURSION IN STOCHASTIC WORLD

- For a while, stop with the above deterministic considerations
- Choose a Brownian motion $(W_t)_{t \geq 0}$ on $\mathbb{R}^d$
- Lift it to $\mathcal{P}_2(\mathbb{R}^d)$ as follows
- $$t \to \mathbb{B}_t^m := \left(\mathbf{id} + \sqrt{2} W_t\right)_{\#} m$$
- Here **id** is the identity map and so, $\mathbb{B}^m$ is a stochastic motion in $\mathcal{P}_2(\mathbb{R}^d)$, starting at m

EXCURSION IN STOCHASTIC WORLD

- For a while, stop with the above deterministic considerations
- Choose a Brownian motion $(W_t)_{t\geq 0}$ on $\mathbb{R}^d$
- Lift it to $\mathcal{P}_2(\mathbb{R}^d)$ as follows
- $$t \to \mathbb{B}_t^m := \left(\mathbf{id} + \sqrt{2}W_t\right)_{\#} m$$
- Here **id** is the identity map and so, $\mathbb{B}^m$ is a stochastic motion in $\mathcal{P}_2(\mathbb{R}^d)$, starting at m
-

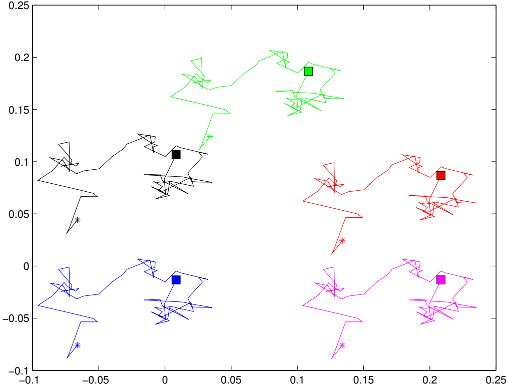
- For a while, stop with the above deterministic considerations
- Choose a Brownian motion $(W_t)_{t\geq 0}$ on $\mathbb{R}^d$
- Lift it to $\mathcal{P}_2(\mathbb{R}^d)$ as follows
- $$t \to \mathbb{B}_t^m := \left(\mathbf{id} + \sqrt{2}W_t\right)_{\#} m$$
- Here **id** is the identity map and so, $\mathbb{B}^m$ is a stochastic motion in $\mathcal{P}_2(\mathbb{R}^d)$, starting at m
-
- For example

$$\mathbb{B}_t^m = \frac{1}{5}\sum_{i=1}^{5} \delta_{x_i+\sqrt{2}W_t}$$

is a population of 5 players evolving with a common noise

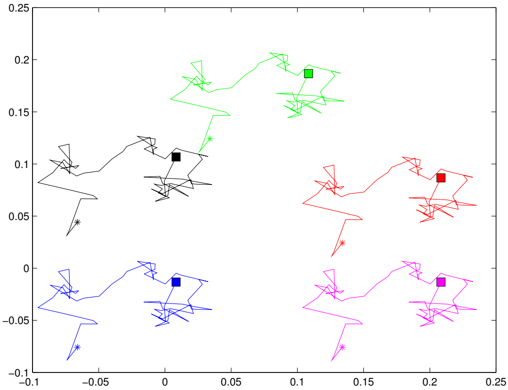
- The Brownian motion is represented in the figure

- The Brownian motion is represented in the figure



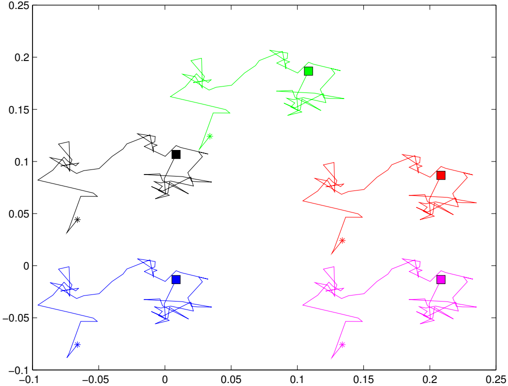
-

- The Brownian motion is represented in the figure



- Notice all the point masses perform a coherent/identical Brownian motion.

- The Brownian motion is represented in the figure



- Notice all the point masses perform a coherent/identical Brownian motion.

Linking geometry to stochastic

- Let $\{E_j\}_{j=1}^d$ be an orthonormal basis in $\mathbb{R}^d$. We define the partial Wasserstein Laplacian at m as the trace:

$$\triangle_w G[m] = \sum_{j=1}^{d} \mathrm{Hess}\ G[m](E_j, E_j)$$

Linking geometry to stochastic

- Let $\{E_j\}_{j=1}^d$ be an orthonormal basis in $\mathbb{R}^d$. We define the partial Wasserstein Laplacian at m as the trace:

$$\triangle_w G[m] = \sum_{j=1}^{d} \mathrm{Hess}\ G[m](E_j, E_j)$$

- The first argument supporting the fact that we have the right definition is

$$\triangle_w G[m] = \lim_{t \to 0} \frac{\mathbb{E}\big(G(\mathbb{B}_t^m)\big) - G(m)}{t}$$

Linking geometry to stochastic

- Let $\{E_j\}_{j=1}^d$ be an orthonormal basis in $\mathbb{R}^d$. We define the partial Wasserstein Laplacian at m as the trace:

$$\triangle_w G[m] = \sum_{j=1}^{d} \text{Hess } G[m](E_j, E_j)$$

- The first argument supporting the fact that we have the right definition is

$$\triangle_w G[m] = \lim_{t \to 0} \frac{\mathbb{E}\big(G(\mathbb{B}_t^m)\big) - G(m)}{t}$$

- Another argument related to Feynman–Kac: set

$$U(t, m) := E\big(G(\mathbb{B}_t^m)\big)$$

to discover that

Linking geometry to stochastic

- Let $\{E_j\}_{j=1}^d$ be an orthonormal basis in $\mathbb{R}^d$. We define the partial Wasserstein Laplacian at m as the trace:

$$\triangle_w G[m] = \sum_{j=1}^{d} \mathrm{Hess}\ G[m](E_j, E_j)$$

- The first argument supporting the fact that we have the right definition is

$$\triangle_w G[m] = \lim_{t \to 0} \frac{\mathbb{E}\big(G(\mathbb{B}_t^m)\big) - G(m)}{t}$$

- Another argument related to Feynman–Kac: set

$$U(t, m) := E\Big(G(\mathbb{B}_t^m)\Big)$$

to discover that

- $$\partial_t U = \triangle_w U, \qquad U(0, \cdot) = G$$

Example: system of n particles

- Use the equation

$$\partial_t U = \triangle_w U, \qquad U(0, \cdot) = G$$

Example: system of n particles

- Use the equation

$$\partial_t U = \triangle_w U, \qquad U(0, \cdot) = G$$

- Set

$$G^n(x_1, \cdots, x_n) = G\Big(\frac{1}{n}\sum_{j=1}^{n} \delta_{x_j}\Big).$$

and

$$U\Big(t, \frac{1}{n}\sum_{j=1}^{n} \delta_{x_j}\Big) = U^n(t, x_1, \cdots, x_n)$$

Example: system of n particles

- Use the equation

$$\partial_t U = \triangle_w U, \qquad U(0,\cdot) = G$$

- Set

$$G^n(x_1,\cdots,x_n) = G\Big(\frac{1}{n}\sum_{j=1}^n \delta_{x_j}\Big).$$

and

$$U\Big(t, \frac{1}{n}\sum_{j=1}^n \delta_{x_j}\Big) = U^n(t, x_1,\cdots,x_n)$$

- Then U^n satisfies the PDE

$$\partial_t U^n = \sum_{j,k=1}^n \mathrm{Trace}\,\Big(\nabla_{x_j x_k} U^n\Big), \qquad U^n(0,\cdot) = G^n$$

$n \to \infty$

- Observe the decomposition

$$\sum_{j,k=1}^{n} \mathrm{Trace}\left(\nabla_{x_j x_k} G^n\right) = \sum_{j}^{n} \triangle_{x_j} G^n + \sum_{j \neq k}^{n} \mathrm{Trace}\left(\nabla_{x_j x_k} G^n\right)$$

$n \to \infty$

- Observe the decomposition

$$\sum_{j,k=1}^{n} \mathrm{Trace}\left(\nabla_{x_j x_k} G^n\right) = \sum_{j}^{n} \triangle_{x_j} G^n + \sum_{j \neq k}^{n} \mathrm{Trace}\left(\nabla_{x_j x_k} G^n\right)$$

- Let $\nabla_w G$ and $\nabla_{ww} G$ be the first and second order Wasserstein gradients.

$n \to \infty$

- Observe the decomposition

$$\sum_{j,k=1}^{n} \mathrm{Trace}\left(\nabla_{x_j x_k} G^n\right) = \sum_{j}^{n} \triangle_{x_j} G^n + \sum_{j \neq k}^{n} \mathrm{Trace}\left(\nabla_{x_j x_k} G^n\right)$$

- Let $\nabla_w G$ and $\nabla_{ww} G$ be the first and second order Wasserstein gradients.
- We have

$$\sum_{j=1}^{n} \triangle_{x_j} G^n \to \int_{\mathbb{R}^d} \mathrm{div}_x \left(\nabla_w G[m](x)\right) m(dx).$$

if

$$\frac{1}{n} \sum_{j=1}^{n} \delta_{x_j} \to m$$

- Observe the decomposition

$$\sum_{j,k=1}^{n} \mathrm{Trace}\left(\nabla_{x_j x_k} G^n\right) = \sum_{j}^{n} \triangle_{x_j} G^n + \sum_{j\neq k}^{n} \mathrm{Trace}\left(\nabla_{x_j x_k} G^n\right)$$

- Let $\nabla_w G$ and $\nabla_{ww} G$ be the first and second order Wasserstein gradients.
- We have

$$\sum_{j=1}^{n} \triangle_{x_j} G^n \to \int_{\mathbb{R}^d} \mathrm{div}_x\big(\nabla_w G[m](x)\big)\, m(dx).$$

if

$$\frac{1}{n} \sum_{j=1}^{n} \delta_{x_j} \to m$$

- We have

$$\sum_{j\neq k}^{n} \mathrm{Trace}\left(\nabla_{x_j x_k} G^n\right) \to \int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(\nabla_w^2 G[m](x,a)\Big)\, m(dx) m(da)$$

Partial Laplacian

- Decompose $\triangle_w$ into the sum of two operators:

Partial Laplacian

- Decompose $\triangle_w$ into the sum of two operators:
-

$$\triangle_w G[m] = O(G)(m) + \int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(\nabla_w^2 G[m](x,a)\Big) m(dx)m(da)$$

where

$$O(G)(m) := \int_{\mathbb{R}^d} \mathrm{div}_x\big(\nabla_w G[m](x)\big) m(dx).$$

Partial Laplacian

- Decompose $\triangle_w$ into the sum of two operators:
-

$$\triangle_w G[m] = O(G)(m) + \int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(\nabla_w^2 G[m](x,a)\Big) m(dx)m(da)$$

where

$$O(G)(m) := \int_{\mathbb{R}^d} \mathrm{div}_x\big(\nabla_w G[m](x)\big) m(dx).$$

- Eigenvalues of $-\triangle_w$ in Fourier

$$\lambda_k^2 := 4\pi^2 \Big\| \sum_{j=1}^d \xi_j \Big\|^2 = 4\pi^2 \sum_{j=1}^d \|\xi_j\|^2 - 4\pi^2 \sum_{j \neq k}^d \xi_j \cdot \xi_k$$

Partial Laplacian

- Decompose $\triangle_w$ into the sum of two operators:
- $$\triangle_w G[m] = O(G)(m) + \int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(\nabla_w^2 G[m](x,a)\Big) m(dx)m(da)$$

  where

  $$O(G)(m) := \int_{\mathbb{R}^d} \mathrm{div}_x\big(\nabla_w G[m](x)\big) m(dx).$$
- Eigenvalues of $-\triangle_w$ in Fourier

  $$\lambda_k^2 := 4\pi^2 \Big\| \sum_{j=1}^d \xi_j \Big\|^2 = 4\pi^2 \sum_{j=1}^d \|\xi_j\|^2 - 4\pi^2 \sum_{j\neq k}^d \xi_j \cdot \xi_k$$
- The eigenvalues of O:

  $$-4\pi^2 \sum_{j=1}^d \|\xi_j\|^2$$

  and so, $-O$ is a positive operator responsible for smoothing effects

Partial Laplacian

- There is a perturbation of $\triangle_w$ which is uniformly elliptic:

Partial Laplacian

- There is a perturbation of $\triangle_w$ which is uniformly elliptic:
- Set

$$\triangle_{w,\epsilon} G[m] = (1+\epsilon) O(G)(m) + \int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(\nabla_w^2 G[m](x,a)\Big) m(dx) m(da)$$

Partial Laplacian

- There is a perturbation of $\triangle_w$ which is uniformly elliptic:
- Set

$$\triangle_{w,\epsilon} G[m] = (1+\epsilon)O(G)(m) + \int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(\nabla_w^2 G[m](x,a)\Big) m(dx) m(da)$$

- In other words

$$\triangle_{w,\epsilon} = \triangle_w + \epsilon O$$

Partial Laplacian

- There is a perturbation of $\triangle_w$ which is uniformly elliptic:
- Set

$$\triangle_{w,\epsilon} G[m] = (1+\epsilon) O(G)(m) + \int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(\nabla_w^2 G[m](x,a)\Big) m(dx) m(da)$$

- In other words

$$\triangle_{w,\epsilon} = \triangle_w + \epsilon O$$

- $\triangle_{w,\epsilon}$ has a better smoothing effect

Partial Laplacian

- There is a perturbation of $\triangle_w$ which is uniformly elliptic:
- Set

$$\triangle_{w,\epsilon} G[m] = (1+\epsilon) O(G)(m) + \int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(\nabla_w^2 G[m](x,a)\Big) m(dx) m(da)$$

- In other words

$$\triangle_{w,\epsilon} = \triangle_w + \epsilon O$$

- $\triangle_{w,\epsilon}$ has a better smoothing effect
- Eigenvalues of $-\triangle_{w,\epsilon}$ in Fourier are

$$\lambda_{k,\epsilon}^2 := \epsilon 4\pi^2 \sum_{j=1}^d \|\xi_j\|^2 + 4\pi^2 \Big\|\sum_{j=1}^d \xi_j\Big\|^2 \geq \epsilon 4\pi^2 \sum_{j=1}^d \|\xi_j\|^2$$

Partial Laplacian

- There is a perturbation of $\triangle_w$ which is uniformly elliptic:
- Set

$$\triangle_{w,\epsilon} G[m] = (1+\epsilon) O(G)(m) + \int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(\nabla_w^2 G[m](x,a)\Big) m(dx) m(da)$$

- In other words

$$\triangle_{w,\epsilon} = \triangle_w + \epsilon O$$

- $\triangle_{w,\epsilon}$ has a better smoothing effect
- Eigenvalues of $-\triangle_{w,\epsilon}$ in Fourier are

$$\lambda_{k,\epsilon}^2 := \epsilon 4\pi^2 \sum_{j=1}^{d} \|\xi_j\|^2 + 4\pi^2 \Big\|\sum_{j=1}^{d} \xi_j\Big\|^2 \geq \epsilon 4\pi^2 \sum_{j=1}^{d} \|\xi_j\|^2$$

-

Special functions

- Given $\xi := (\xi_1, \cdots, \xi_k) \in \mathbb{R}^{kd} := \mathbb{R}^d \times \cdots \times \mathbb{R}^d$, set

$$F_\xi^k[m] := \int_{\mathbb{R}^d \times \cdots \times \mathbb{R}^d} \exp\Big(-2\pi i \langle \xi, x \rangle\Big) m(dx_1) \cdots m(dx_k)$$

Special functions

- Given $\xi := (\xi_1, \cdots, \xi_k) \in \mathbb{R}^{kd} := \mathbb{R}^d \times \cdots \times \mathbb{R}^d$, set

$$F_\xi^k[m] := \int_{\mathbb{R}^d \times \cdots \times \mathbb{R}^d} \exp\Big(-2\pi i \langle \xi, x \rangle\Big) m(dx_1) \cdots m(dx_k)$$

- $F_\xi^k[\cdot]$ are eigenfunctions of $\triangle_w$

Special functions

- Given $\xi := (\xi_1, \cdots, \xi_k) \in \mathbb{R}^{kd} := \mathbb{R}^d \times \cdots \times \mathbb{R}^d$, set

$$F_\xi^k[m] := \int_{\mathbb{R}^d \times \cdots \times \mathbb{R}^d} \exp\Big(-2\pi i \langle \xi, x \rangle\Big) m(dx_1) \cdots m(dx_k)$$

- $F_\xi^k[\cdot]$ are eigenfunctions of $\triangle_w$
- The superpositions of the $F_\xi^k[\cdot]$ yield Sobolev functions

Special functions

- Given $\xi := (\xi_1, \cdots, \xi_k) \in \mathbb{R}^{kd} := \mathbb{R}^d \times \cdots \times \mathbb{R}^d$, set

$$F_\xi^k[m] := \int_{\mathbb{R}^d \times \cdots \times \mathbb{R}^d} \exp\Big(-2\pi i \langle \xi, x \rangle\Big) m(dx_1) \cdots m(dx_k)$$

- $F_\xi^k[\cdot]$ are eigenfunctions of $\triangle_w$
- The superpositions of the $F_\xi^k[\cdot]$ yield Sobolev functions
-

$$F[m] := \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^d \times \cdots \times \mathbb{R}^d} a_k(\xi) F_\xi^k[m] d\xi$$

  on

$$\mathcal{M} := \mathcal{P}_2(\mathbb{R}^d)$$

  when appropriate conditions are imposed on $(a_k)_k$.

DATA

- Say $F \in H^s(\mathcal{M})$ if

$$\sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^{dk}} |a_k(\xi)|^2 \left(1 + \lambda_k^2(\xi)\right)^s d\xi < \infty.$$

DATA

- Say $F \in H^s(\mathcal{M})$ if

$$\sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^{dk}} |a_k(\xi)|^2 \left(1 + \lambda_k^2(\xi)\right)^s d\xi < \infty.$$

- The series converges uniformly if there are $C, \delta > 0$ such that

$$|a_k(\xi)| \leq \frac{Ck!}{k^\delta}$$

DATA

- Say $F \in H^s(\mathcal{M})$ if

$$\sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^{dk}} |a_k(\xi)|^2 \big(1 + \lambda_k^2(\xi)\big)^s d\xi < \infty.$$

- The series converges uniformly if there are $C, \delta > 0$ such that

$$|a_k(\xi)| \leq \frac{Ck!}{k^\delta}$$

- Theorem: If $F \in H^s(\mathcal{M})$ setting

$$U(t, m) := \mathbb{E}\Big(F\big(\mathbb{B}_t^m\big)\Big)$$

DATA

- Say $F \in H^s(\mathcal{M})$ if

$$\sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^{dk}} |a_k(\xi)|^2 \big(1 + \lambda_k^2(\xi)\big)^s d\xi < \infty.$$

- The series converges uniformly if there are $C, \delta > 0$ such that

$$|a_k(\xi)| \leq \frac{Ck!}{k^\delta}$$

- Theorem: If $F \in H^s(\mathcal{M})$ setting

$$U(t, m) := \mathbb{E}\Big(F\big(\mathbb{B}_t^m\big)\Big)$$

- we have

$$\partial_t U = \triangle_w U, \quad U(0, \cdot) = F.$$

DATA

- Say $F \in H^s(\mathcal{M})$ if

$$\sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^{dk}} |a_k(\xi)|^2 \big(1 + \lambda_k^2(\xi)\big)^s d\xi < \infty.$$

- The series converges uniformly if there are $C, \delta > 0$ such that

$$|a_k(\xi)| \leq \frac{Ck!}{k^\delta}$$

- Theorem: If $F \in H^s(\mathcal{M})$ setting

$$U(t, m) := \mathbb{E}\Big(F\big(\mathbb{B}_t^m\big)\Big)$$

- we have

$$\partial_t U = \triangle_w U, \quad U(0, \cdot) = F.$$

- For any $r \geq s$, $U(t, \cdot) \in H^r(\mathcal{M})$

Brief review on conservation laws

- Let us go back to the finite dimensional setting.

Brief review on conservation laws

- Let us go back to the finite dimensional setting.
- Let

$$H \in C^3(\mathbb{R}^{2d}) \quad \text{be such that} \quad H(q, \cdot) \quad \text{is strictly convex}$$

Brief review on conservation laws

- Let us go back to the finite dimensional setting.
- Let

$$H \in C^3(\mathbb{R}^{2d}) \quad \text{be such that} \quad H(q,\cdot) \quad \text{is strictly convex}$$

- Consider the Hamilton–Jacobi equation

$$\partial_t A + H(q, \nabla A) = 0 \quad \text{on} \quad (0,\infty) \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d$$

Brief review on conservation laws

- Let us go back to the finite dimensional setting.
- Let

$$H \in C^3(\mathbb{R}^{2d}) \quad \text{be such that} \quad H(q,\cdot) \quad \text{is strictly convex}$$

- Consider the Hamilton–Jacobi equation

$$\partial_t A + H(q, \nabla A) = 0 \quad \text{on} \quad (0,\infty) \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d$$

- Set $\mathbf{a} := \nabla A$. It is well–known that by differentiation one obtains the conservation laws (when $\epsilon = 0$)

$$\partial_t \mathbf{a} + \nabla\Big(H(q, \mathbf{a})\Big) = \epsilon \triangle \mathbf{a} \quad \text{on} \quad (0,\infty) \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d$$

Brief review on conservation laws

- Let us go back to the finite dimensional setting.
- Let

$$H \in C^3(\mathbb{R}^{2d}) \quad \text{be such that} \quad H(q,\cdot) \quad \text{is strictly convex}$$

- Consider the Hamilton–Jacobi equation

$$\partial_t A + H(q, \nabla A) = 0 \quad \text{on} \quad (0,\infty) \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d$$

- Set $\mathbf{a} := \nabla A$. It is well–known that by differentiation one obtains the conservation laws (when $\epsilon = 0$)

$$\partial_t \mathbf{a} + \nabla \Big(H(q, \mathbf{a}) \Big) = \epsilon \triangle \mathbf{a} \quad \text{on} \quad (0,\infty) \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d$$

- Similarly,

$$\partial_t A + H(q, \nabla A) = \epsilon \triangle A \quad \Longrightarrow \quad \partial_t \mathbf{a} + \nabla \Big(H(q, \mathbf{a}) \Big) = \epsilon \triangle \mathbf{a}$$

Mean Field Game $n = \infty$

- We would like to write the analogue of this reasoning in infinite dimension

Mean Field Game $n = \infty$

- We would like to write the analogue of this reasoning in infinite dimension
- We define the Hamiltonian

$$\mathcal{H}(m, \zeta) := \int_{\mathbb{R}^d} H(q, \zeta(q)m(dq)$$

Mean Field Game $n = \infty$

- We would like to write the analogue of this reasoning in infinite dimension
- We define the Hamiltonian

$$\mathcal{H}(m, \zeta) := \int_{\mathbb{R}^d} H(q, \zeta(q)m(dq)$$

- For a system of n particles

$$m = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \quad \text{represents the position of particles}$$

Mean Field Game $n = \infty$

- We would like to write the analogue of this reasoning in infinite dimension
- We define the Hamiltonian

$$\mathcal{H}(m, \zeta) := \int_{\mathbb{R}^d} H(q, \zeta(q) m(dq)$$

- For a system of n particles

$$m = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \quad \text{represents the position of particles}$$

- The velocity is

$$\mathbf{v} = \sum_{i=1}^{n} \xi_i \, \delta_{x_i}$$

Mean Field Game $n = \infty$

- We would like to write the analogue of this reasoning in infinite dimension
- We define the Hamiltonian

$$\mathcal{H}(m, \zeta) := \int_{\mathbb{R}^d} H(q, \zeta(q) m(dq)$$

- For a system of n particles

$$m = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \quad \text{represents the position of particles}$$

- The velocity is

$$\mathbf{v} = \sum_{i=1}^{n} \xi_i \, \delta_{x_i}$$

- The Hamiltonian restricted to a finite system is

$$\mathcal{H}(m, \mathbf{v}) = \frac{1}{n} H(x_i, \xi_i)$$

- Consider the Hamilton–Jacobi equation: Find aaa

$$U^n : (0, T) \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to \mathbb{R}$$

such that

$$\partial_t \mathcal{U}^n(t, x) + \mathcal{H}\big(m^x, \nabla_w \mathcal{U}^n(t, x)\big) = \epsilon_1 \triangle_n \mathcal{U}^n(t, x) + \epsilon_2 O_n(\mathcal{U}^n(t, x))$$

- Consider the Hamilton–Jacobi equation: Find aaa

$$U^n : (0, T) \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to \mathbb{R}$$

such that

$$\partial_t \mathcal{U}^n(t, x) + \mathcal{H}\big(m^x, \nabla_w \mathcal{U}^n(t, x)\big) = \epsilon_1 \triangle_n \mathcal{U}^n(t, x) + \epsilon_2 O_n(\mathcal{U}^n(t, x))$$

- Here,

$$\triangle_n = \sum_j^n \triangle_{x_j} + \sum_{j \neq k}^n \mathrm{Trace}\left(\nabla_{x_j x_k}\right), \qquad O_n := \sum_j^n \triangle_{x_j}$$

- Consider the Hamilton–Jacobi equation: Find aaa

$$U^n : (0, T) \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to \mathbb{R}$$

such that

$$\partial_t \mathcal{U}^n(t, x) + \mathcal{H}\big(m^x, \nabla_w \mathcal{U}^n(t, x)\big) = \epsilon_1 \triangle_n \mathcal{U}^n(t, x) + \epsilon_2 O_n(\mathcal{U}^n(t, x))$$

- Here,

$$\triangle_n = \sum_j^n \triangle_{x_j} + \sum_{j \neq k}^n \mathrm{Trace}\left(\nabla_{x_j x_k}\right), \qquad O_n := \sum_j^n \triangle_{x_j}$$

- Let n tend to ∞ to obtain an infinite dimensional equation

Mean Field Game $n = \infty$

- Coming back to the infinite dimension

Mean Field Game $n = \infty$

- Coming back to the infinite dimension
- Recall the Hamiltonian

$$\mathcal{H}(m, \zeta) := \int_{\mathbb{R}^d} H(q, \zeta(q) m(dq)$$

Mean Field Game $n = \infty$

- Coming back to the infinite dimension
- Recall the Hamiltonian

$$\mathcal{H}(m, \zeta) := \int_{\mathbb{R}^d} H(q, \zeta(q)m(dq)$$

- Consider on $\mathcal{P}_2(\mathbb{R}^d)$, the Hamilton–Jacobi equation:

$$\partial_t \mathcal{U} + \mathcal{H}(m, \nabla_w \mathcal{U}) = \epsilon_1 \triangle_w \mathcal{U} + \epsilon_2 O(\mathcal{U}) \quad \text{on} \quad (0, T) \times \mathcal{P}_2(\mathbb{R}^d) \quad (2)$$

Mean Field Game $n = \infty$

- Coming back to the infinite dimension
- Recall the Hamiltonian

$$\mathcal{H}(m, \zeta) := \int_{\mathbb{R}^d} H(q, \zeta(q) m(dq)$$

- Consider on $\mathcal{P}_2(\mathbb{R}^d)$, the Hamilton–Jacobi equation:

$$\partial_t \mathcal{U} + \mathcal{H}(m, \nabla_w \mathcal{U}) = \epsilon_1 \triangle_w \mathcal{U} + \epsilon_2 O(\mathcal{U}) \quad \text{on} \quad (0, T) \times \mathcal{P}_2(\mathbb{R}^d) \ (2)$$

- Set $u := \delta_{L^2} U$ and apply δ_{L^2} to (2)

Mean Field Game $n = \infty$

- Coming back to the infinite dimension
- Recall the Hamiltonian

$$\mathcal{H}(m, \zeta) := \int_{\mathbb{R}^d} H(q, \zeta(q) m(dq)$$

- Consider on $\mathcal{P}_2(\mathbb{R}^d)$, the Hamilton–Jacobi equation:

$$\partial_t \mathcal{U} + \mathcal{H}(m, \nabla_w \mathcal{U}) = \epsilon_1 \triangle_w \mathcal{U} + \epsilon_2 O(\mathcal{U}) \quad \text{on} \quad (0, T) \times \mathcal{P}_2(\mathbb{R}^d) \ (2)$$

- Set $u := \delta_{L^2} U$ and apply δ_{L^2} to (2)
- We obtain the non–local equation

$$\partial_t u + H(m, \nabla u) + \int_{\mathbb{R}^d} \Big\langle \nabla_w u ; D_p H\big(\cdot, \nabla u\big)\Big\rangle m(dy) =$$

$$\epsilon_1 \Big(\underline{\triangle_w u} + \triangle u + 2 \int_{\mathbb{R}^d} \mathrm{div} \Big[\nabla_w \Big(u[q, m] \Big)(y) \Big] m(dy) \Big) + \epsilon_2 \Big(\underline{O(u)} + \triangle u \Big)$$

Mean Field Game $n = \infty$

- This is an equation introduced in Mean Field Games by PL. Lions

Mean Field Game $n = \infty$

- This is an equation introduced in Mean Field Games by PL. Lions
- Summary on latest pages:

Mean Field Game $n = \infty$

- This is an equation introduced in Mean Field Games by PL. Lions
- Summary on latest pages:
- by differentiating an infinite dimensional Hamilton–Jacobi equation, an equation of local nature, one obtains an infinite dimensional perturbed conservation law system

Mean Field Game $n = \infty$

- This is an equation introduced in Mean Field Games by PL. Lions
- Summary on latest pages:
- by differentiating an infinite dimensional Hamilton–Jacobi equation, an equation of local nature, one obtains an infinite dimensional perturbed conservation law system
- A side lesson we learnt:

Mean Field Game $n = \infty$

- This is an equation introduced in Mean Field Games by PL. Lions
- Summary on latest pages:
- by differentiating an infinite dimensional Hamilton–Jacobi equation, an equation of local nature, one obtains an infinite dimensional perturbed conservation law system
- A side lesson we learnt:
- the difficulties in conservation laws on $\mathbb{R}^d$ will not go awayon $\mathcal{P}_2(\mathbb{R}^d)$

Mean Field Game $n = \infty$

- This is an equation introduced in Mean Field Games by PL. Lions

- Summary on latest pages:
- by differentiating an infinite dimensional Hamilton–Jacobi equation, an equation of local nature, one obtains an infinite dimensional perturbed conservation law system

- A side lesson we learnt:
- the difficulties in conservation laws on $\mathbb{R}^d$ will not go awayon $\mathcal{P}_2(\mathbb{R}^d)$
-

CONCLUSIONS

- I hope I have convinced you:
  - We have the right concept of Wasserstein Laplacian
  - the Wasserstein Laplacian is important and applicable

CONCLUSIONS

- I hope I have convinced you:
  - We have the right concept of Wasserstein Laplacian
  - the Wasserstein Laplacian is important and applicable

- COMPLETES THIS TALK

CONCLUSIONS

- I hope I have convinced you:
  - We have the right concept of Wasserstein Laplacian
  - the Wasserstein Laplacian is important and applicable

- COMPLETES THIS TALK
- Thanks for your attention

CONCLUSIONS

- I hope I have convinced you:
  - We have the right concept of Wasserstein Laplacian
  - the Wasserstein Laplacian is important and applicable
- COMPLETES THIS TALK
- Thanks for your attention
-

CONCLUSIONS

- I hope I have convinced you:
  - We have the right concept of Wasserstein Laplacian
  - the Wasserstein Laplacian is important and applicable

- COMPLETES THIS TALK
- Thanks for your attention
-
- Outstanding open problem:

CONCLUSIONS

- I hope I have convinced you:
  - We have the right concept of Wasserstein Laplacian
  - the Wasserstein Laplacian is important and applicable
- COMPLETES THIS TALK
- Thanks for your attention
-
- Outstanding open problem:
- find the optimal cost for transporting

CONCLUSIONS

- I hope I have convinced you:
  - We have the right concept of Wasserstein Laplacian
  - the Wasserstein Laplacian is important and applicable
- COMPLETES THIS TALK
- Thanks for your attention
-
- Outstanding open problem:
- find the optimal cost for transporting

W GAN from deep learning to

-

CONCLUSIONS

- I hope I have convinced you:
  - We have the right concept of Wasserstein Laplacian
  - the Wasserstein Laplacian is important and applicable

- COMPLETES THIS TALK
- Thanks for your attention
-
- Outstanding open problem:
- find the optimal cost for transporting

  W GAN from deep learning to
-

  WILFRID GANGBO in OT
-